The art of queuing — countable and uncountable sets

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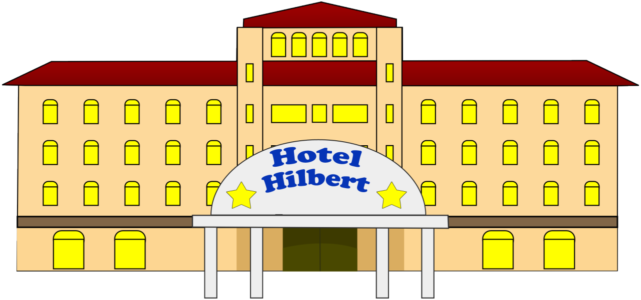
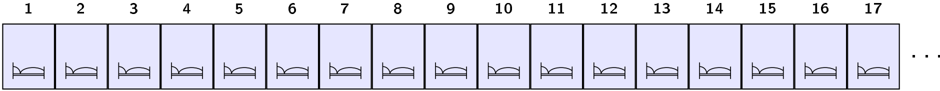
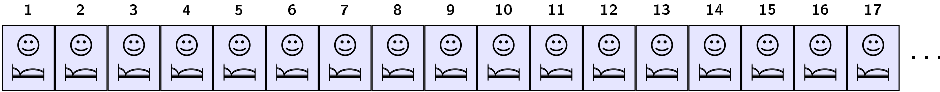


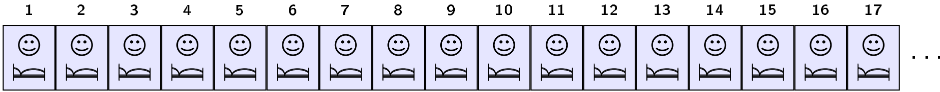
Hilbert's hotel: a big hotel with infinitely many numbered rooms



The hotel is full!

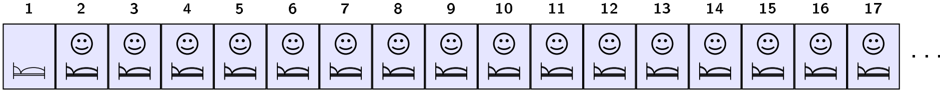


A single traveler arrives.



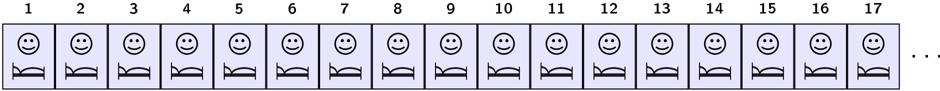
☹?

"Everybody moves to the room with the successive number."

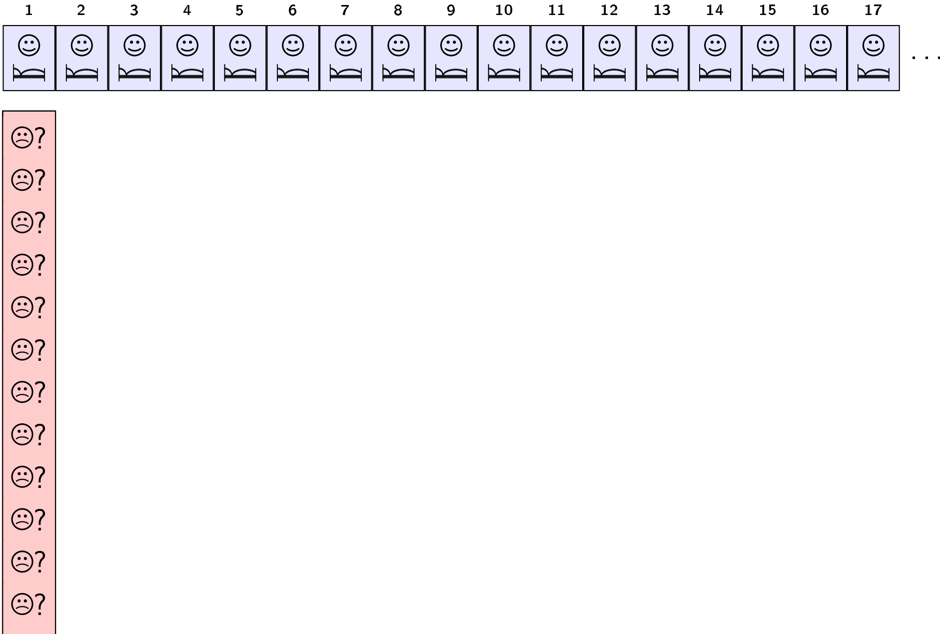


☺!

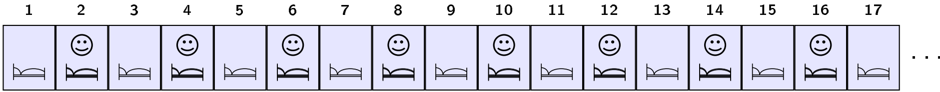
‘Another sock always fits into the suitcase.’

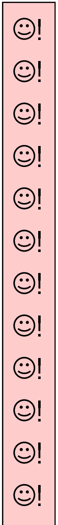


A full bus with a single seat row of infinite length arrives!

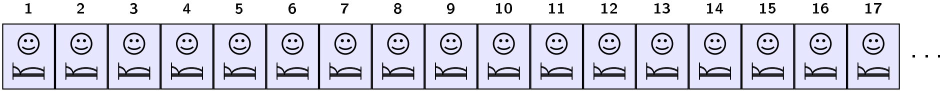


"Everyone gets a new room number which is twice his old one."

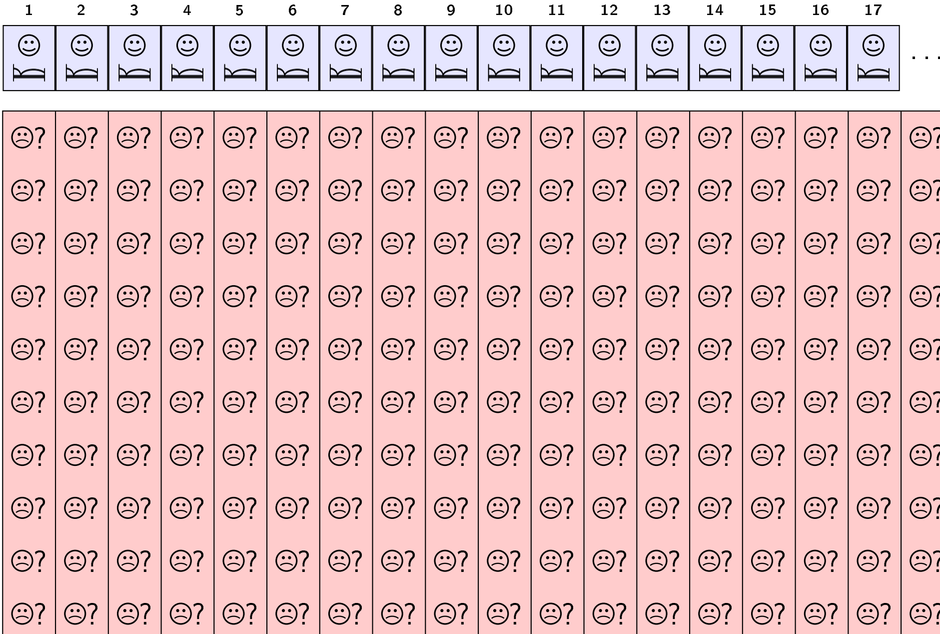




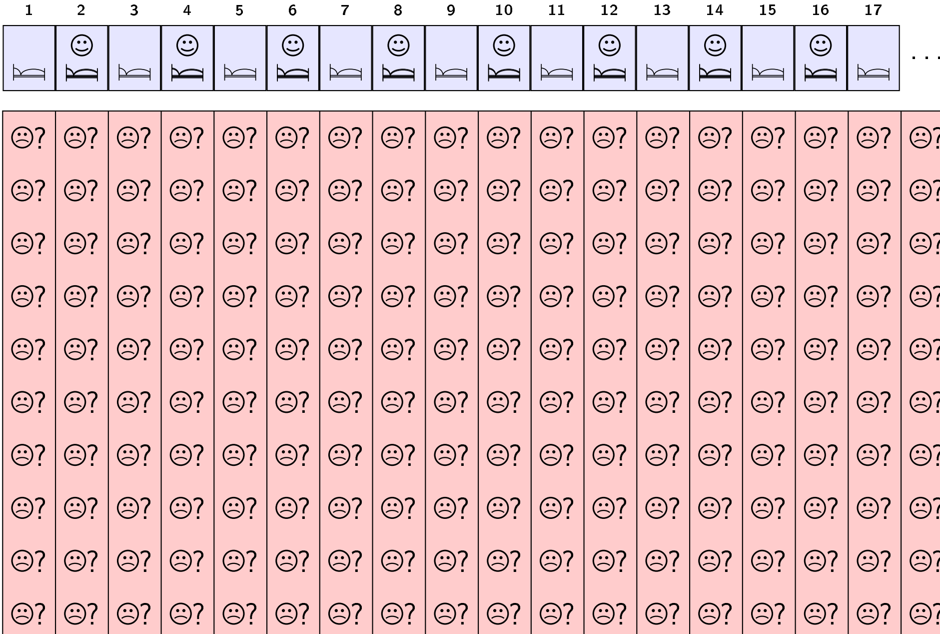
‘It is a big hotel!’



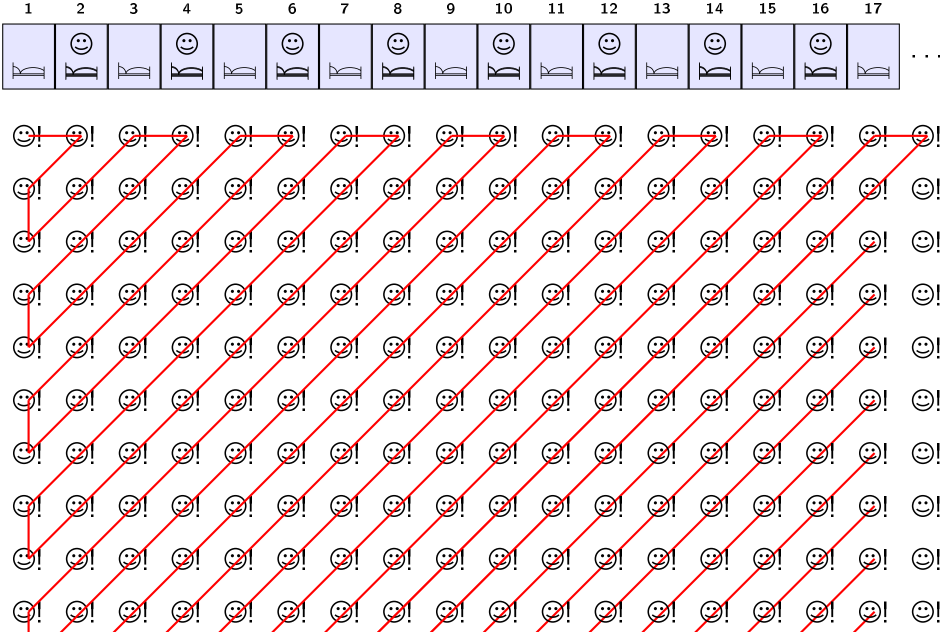
Infinitely many numbered busses of infinite length arrive.



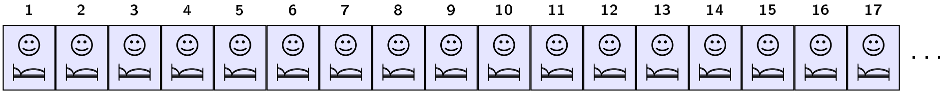
“Everyone gets a new room number which is twice his old one.”



‘It only depends on how you queue.’



'Can this hotel host every traveler group?'



Power sets – the limits of queuing! (Cantor's diagonal method)

Imagine a traveller group of crazy linguists.

- Every traveller wears a T-shirt with a formal language over the alphabet $\Sigma = \{a\}$ printed on it.
- Every formal language over $\Sigma = \{a\}$ is printed on exactly one T-shirt.

Do they all fit into the hotel?

crazy linguists representing the power set of $\{a\}^*$, i.e. $2^{\{a\}^*} = \{L | L \subseteq \{a\}^*\}$

Power sets – the limits of queuing! (Cantor's diagonal method)

a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}	a^{16}	a^{17}	a^{18}	a^{19}	a^{20}
0	0	1	0	1	1	1	0	1	0	0	1	1	1	1	1	1	0	1	0	1
1	1	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1	1	0	1	0
1	1	1	0	1	0	1	0	0	1	0	1	1	1	0	0	1	1	1	1	1
1	0	1	1	1	1	1	0	1	1	1	1	0	1	1	1	1	1	0	1	1
1	1	0	1	1	1	1	1	0	1	1	0	1	0	1	1	1	1	1	1	0
1	0	1	1	0	1	1	0	1	1	1	0	1	1	1	1	0	1	1	1	1
1	0	1	0	1	1	1	0	1	0	0	1	1	0	1	0	1	1	1	1	1
1	0	1	1	1	1	1	0	1	1	1	0	1	1	1	1	0	1	1	1	1
0	1	1	0	1	1	0	1	1	0	0	1	0	1	0	1	1	0	1	0	1
1	1	0	1	0	1	1	1	0	1	1	0	1	1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	1	1
1	1	1	0	1	1	1	1	1	0	0	0	0	1	0	1	1	1	0	1	1
1	0	1	0	1	0	1	0	1	0	0	0	1	1	1	0	1	1	1	1	1
1	1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	0	1	0	1	1
1	1	0	0	1	1	1	1	0	1	1	0	1	1	1	0	1	0	1	1	1
1	0	1	1	1	1	0	1	1	1	1	1	0	1	1	0	0	1	1	1	0

Power sets – the limits of queuing! (Cantor's diagonal method)

a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}	a^{16}	a^{17}	a^{18}	a^{19}	a^{20}
0	0	1	0	1	1	1	0	1	0	0	1	1	1	1	1	1	0	1	0	1
1	**1**	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1	1	0	1	0
1	1	**1**	0	1	0	1	0	0	1	0	1	1	1	0	0	1	1	1	1	1
1	0	1	**1**	1	1	1	0	1	1	1	1	0	1	1	1	1	1	0	1	1
1	1	0	1	**1**	1	1	1	0	1	1	0	1	0	1	1	1	1	1	1	0
1	0	1	1	0	**1**	1	0	1	1	1	0	1	1	1	1	0	1	1	1	1
1	0	1	0	1	1	**1**	0	1	0	0	1	1	0	1	0	1	1	1	1	1
1	0	1	1	1	1	1	**0**	1	1	1	0	1	1	1	1	0	1	1	1	1
0	1	1	0	1	1	0	1	**1**	0	0	1	0	1	0	1	1	0	1	0	1
1	1	0	1	0	1	1	1	0	**1**	1	0	1	1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1	1	1	**1**	1	0	1	1	1	1	0	1	1	1
1	1	1	0	1	1	1	1	1	0	0	**0**	0	1	0	1	1	1	0	1	1
1	0	1	0	1	0	1	0	1	0	0	0	**1**	1	1	0	1	1	1	1	1
1	1	0	1	1	1	0	1	1	0	0	1	0	**1**	0	1	0	1	0	1	1
1	1	0	0	1	1	1	1	0	1	1	0	1	1	**1**	0	1	0	1	1	1
1	0	1	1	1	1	0	1	1	1	1	1	0	1	1	**0**	0	1	1	1	0

Power sets – the limits of queuing! (Cantor's diagonal method)

a^0	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}	a^{16}	a^{17}	a^{18}	a^{19}	a^{20}
1	0	1	0	1	1	1	0	1	0	0	1	1	1	1	1	1	0	1	0	1
1	**0**	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1	1	0	1	0
1	1	**0**	0	1	0	1	0	0	1	0	1	1	1	0	0	1	1	1	1	1
1	0	1	**0**	1	1	1	0	1	1	1	1	0	1	1	1	1	1	0	1	1
1	1	0	1	**0**	1	1	1	0	1	1	0	1	0	1	1	1	1	1	1	0
1	0	1	1	0	**0**	1	0	1	1	1	0	1	1	1	1	0	1	1	1	1
1	0	1	0	1	1	**0**	0	1	0	0	1	1	0	1	0	1	1	1	1	1
1	0	1	1	1	1	1	**1**	1	1	1	0	1	1	1	1	0	1	1	1	1
0	1	1	0	1	1	0	1	**0**	0	0	1	0	1	0	1	1	0	1	0	1
1	1	0	1	0	1	1	1	0	**0**	1	0	1	1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1	1	1	**0**	1	0	1	1	1	1	0	1	1	1
1	1	1	0	1	1	1	1	1	0	0	**1**	0	1	0	1	1	1	0	1	1
1	0	1	0	1	0	1	0	1	0	0	0	**0**	1	1	0	1	1	1	1	1
1	1	0	1	1	1	0	1	1	0	0	1	0	**0**	0	1	0	1	0	1	1
1	1	0	0	1	1	1	1	0	1	1	0	1	1	**0**	0	1	0	1	1	1
1	0	1	1	1	1	0	1	1	1	1	1	0	1	1	**1**	0	1	1	1	0

Some terminology

- A set S is **finite** if there is no proper subset $S' \subsetneq S$ for which a bijection (1-1 correspondence) $f : S' \to S$ exists.
- A set is **infinite** if it is not finite.
- A set S is **countable** if there exists a bijection $f : S \to \mathbb{N}$.
- A set is **uncountable** if it is infinite, but not countable.