Mereology

CLASSICAL EXTENSIONAL MEREOLOGY

- Mereology
- Core axioms and concepts
  - parthood
  - sum
- Higher order properties:
  - cumulativity
  - divisivity (aka divisiveness)
  - atomicity

CLASSICAL EXTENSIONAL MEREOLOGY

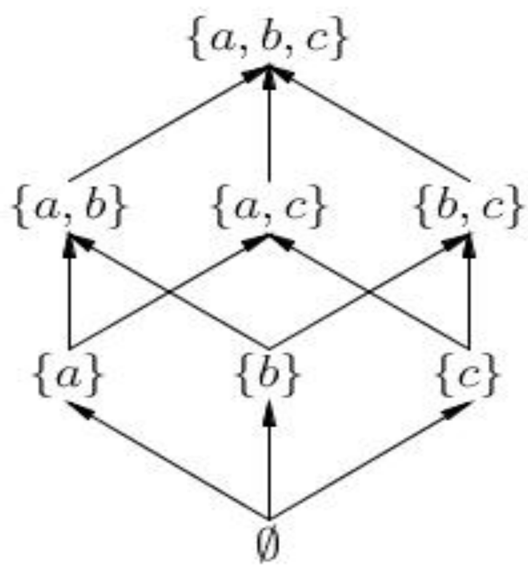
- Mereology
  - is the theory of parthood
  - derived from the Greek *μέρος* (meros) meaning , "part" (also "portion", "segment")
  - origins: the Pre-Socratics (6th and 5th century BC, see Varzi 2011), Leśniewski (1916), Leonard & Goodman (1940) and Goodman (1951)
  - formalized by means of mathematical structures: namely, Boolean algebras.

- In the Boolean algebra
  - the values of variables are the truth values (true, false),
  - the main operations are
    - conjunction (or meet) $\wedge$,
    - disjunction $\vee$, and
    - negation **¬.**

    (In contrast, in elementary algebra, the values of variables are numbers and the main operations are addition and multiplication.)
  - A common way of defining a Boolean algebra is as a lattice structure, a type of algebraic structure [see next slide].

CLASSICAL EXTENSIONAL MEREOLOGY

Boolean lattice of subsets



Is a subset of $\subseteq$

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CLASSICAL EXTENSIONAL MEREOLOGY

- Classical Extensional Mereology (CEM) consists of
  - **THREE AXIOMS** and requires only
  - a **SINGLE PRIMITIVE NOTION** in terms of which the rest of the mereological system can be defined.

- The three basic axioms are given in Lewis (1991) informally as follows:

  - AXIOM 1 (**Unrestricted Composition**): Whenever there are some objects, then there exists a mereological sum of those objects.
  - AXIOM 2 (**Uniqueness of Composition**): It never happens that the same objects have two different mereological sums.
  - AXIOM 3 (**Transitivity**): If x is part of some part of y, then x is part of y.

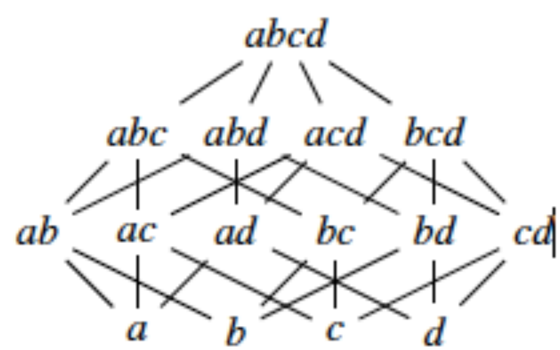
- The single primitive can be chosen to be
  - **proper parthood** $<$,
  - **proper-or-improper parthood** $\leq$,
  - **sum** $\oplus$,
  - **overlap** $\otimes$
  - **disjointness**.

Other notions are definable in terms of whichever one is taken as primitive.

CLASSICAL EXTENSIONAL MEREOLOGY

- AXIOM 1 (**Unrestricted Composition**): Whenever there are some objects, then there exists a mereological sum of those objects.
- Example: Suppose the entire universe consists of
  - Ann (*a*),
  - Bill (*b*),
  - one car (*c*) and
  - one dog (*d*).

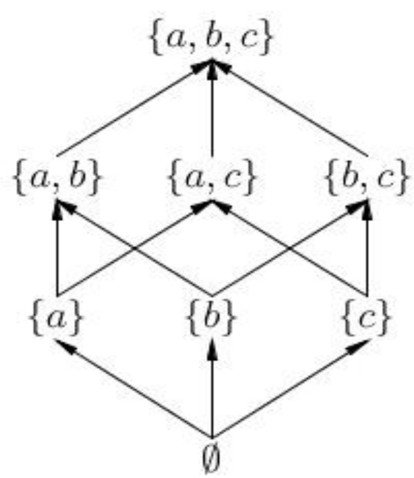
then we need to represent not only these four entities (at the bottom of the lattice) but all of their combinations, among which is Ann together with Bill which corresponds to the meaning of the conjunction *Ann and Bill*:

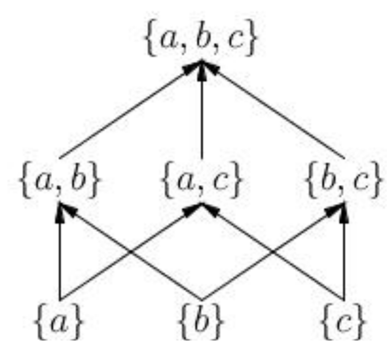


CLASSICAL EXTENSIONAL MEREOLOGY

No "null individual"

- The standard versions of CEM used in philosophy and semantic theory restrict the admissible algebraic structures to those that have no “null individual”, i.e., an individual which belongs to all other individuals in the way that the empty set is a member of all other sets in set theory.
- The existence of such a null individual is taken to be counterintuitive.
- Consequently, the structures that are assumed are a special type of lattice, a **SEMILATTICE,** an UPPER SEMILATTICE. The "semi-" indicates that the structure is closed under only one operation, here **sum** operation.



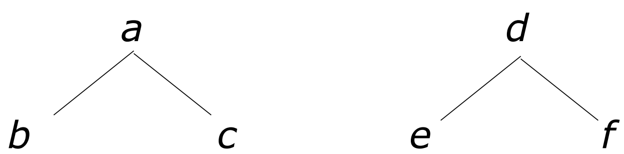


SEMILATTICE: Boolean algebra structure with the bottom null element removed

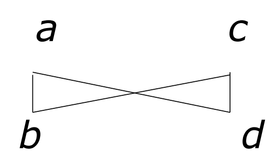
CLASSICAL EXTENSIONAL MEREOLOGY

- AXIOM 2 (**Uniqueness of Composition**) excludes (1) and (2), because not every two elements have a unique sum.

(1)

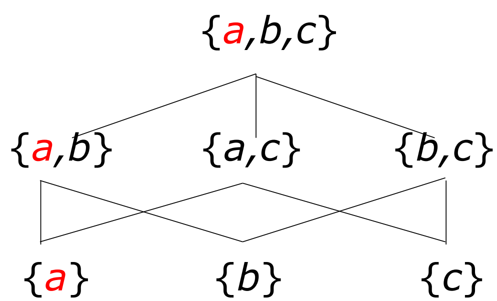


(2)



- AXIOM 3 (**Transitivity**): {*a*} is a part of {*a*,*b*,*c*}, because it is a part of one its parts.

(3)



CLASSICAL EXTENSIONAL MEREOLOGY

- The single primitive can be chosen to be
  - **proper parthood** $<$,
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  - **overlap** $\otimes$
  - **disjointness**.

  The other notions are definable in terms of whichever one is taken as primitive.

- Most commonly
  - the **part $\leq$ relation** is taken as the primitive notion and the sum operation defined from it (Tarski 1929, 1956), or
  - the **sum $\oplus$ operation** is taken as the primitive notion and the part relation defined from it (e.g., Krifka 1986 and elsewhere).

CLASSICAL EXTENSIONAL MEREOLOGY

The sum operation as the primitive notion and the part relation defined from it.

Krifka (1998, p.199): Definition of a part structure *P*

- $P = \langle U_P, \oplus_P, \leq_P, <_P, \otimes_P \rangle$ is a part structure, iff

  a. 'U_P' is a set of entities: individuals, eventualities and times

  $I_P \cup E_P \cup T_P \subset U_P$

  b. '$\oplus_P$' is a binary **sum operation**, it is a function from $U_P \times U_P$ to U_P. (It is idempotent, commutative, associative:

  $\forall x,y,z \in U_P[x \oplus_P x = x \wedge x \oplus_P y = y \oplus_P x \wedge x \oplus_P (y \oplus_P z) = (x \oplus_P y) \oplus_P z]$

  c. '$\leq_P$' is the **part relation**: $\forall x,y \in U_P\ [x \leq_P y \leftrightarrow x \oplus_P y = y]$

  d. '$<_P$' is the **proper part relation**: $\forall x,y \in U_P\ [x <_P y \leftrightarrow x \leq_P y \wedge x \neq y]$

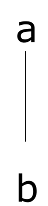
  e. '$\otimes_P$' is the **overlap relation**: $\forall x,y,z \in U_P\ [x \otimes_P y \leftrightarrow \exists z \in U_P[z \leq_P x \wedge z \leq_P y]]$

  f. **remainder principle**: $\forall x,y,z \in U_P\ [x <_P y \rightarrow \exists! z[\neg\ [z \otimes_P x] \wedge z \oplus_P x = y\ ]]$

CLASSICAL EXTENSIONAL MEREOLOGY

- An axiom known as the **REMAINDER PRINCIPLE** or SUPPLEMENTATION is used in order to ensure that the following structures be excluded.

  The object *a* has a solitary proper part *b*:



- **REMAINDER PRINCIPLE**: $\forall x,y,z \in U_P\ [x <_P y \rightarrow \exists! z[\neg\ [z \otimes_P x] \wedge z \oplus_P x = y\ ]]$

  Whenever *x* is a proper part of *y*, there is exactly one "remainder" *z* that does not overlap with *x* such that the sum of *z* and *x* is *y* (Krifka 1998)

  Alternative definition: $\forall x,y,z \in U_P\ [x <_P y \rightarrow \exists z[\neg[z \otimes_P x] \wedge z \leq_P y]$

CLASSICAL EXTENSIONAL MEREOLOGY

The part relation as the primitive notion and the sum operation defined from it.

Tarski (1929, 1956)

- The **"part-of"** relation is reflexive, transitive and antisymmetric:

Axiom of reflexivity: $\forall x[x \leq x]$
Everything is part of itself.

Axiom of transitivity: $\forall x \forall y \forall z[x \leq y \wedge y \leq z \rightarrow x \leq z]$
Any part of any part of a thing is itself part of that thing.

Axiom of antisymmetry: $\forall x \forall y[x \leq y \wedge y \leq x \rightarrow x = y]$
Two distinct things cannot both be part of each other.

CLASSICAL EXTENSIONAL MEREOLOGY

Based on the "part-of" relation $\leq$, we define the relations of proper-part and overlap:

- **proper-part-of relation <**
  The "proper-part-of" relation restricts parthood to nonequal pairs:

  $$x < y =_{def} x \leq y \wedge x \neq y$$

  A proper part of a thing is a part of it that is distinct from it.

  or

  $$x < y =_{def} x \leq y \wedge \neg(y \leq x)$$

  x is a proper part of a thing if it is a part of a thing which itself is not part of x.

- **overlap relation ⊗**

  $$x \otimes y =_{def} \exists z[z \leq x \wedge z \leq y]$$

  Two things overlap if and only if they have a part in common.

CLASSICAL EXTENSIONAL MEREOLOGY

The sum operation $\oplus$

- The classical definition is due to Tarski (1929, 1956). (For other definitions, see Sharvy 1979, 1980, for instance.)

$$\text{sum}(x,P) =\text{def} \ \ \forall y[P(y) \rightarrow y \leq x] \wedge \forall z[z \leq x \rightarrow \exists z'[P(z') \wedge z \otimes z']]$$

  - A sum of a set P is a thing that contains everything in *P* and whose parts each overlap with something in P.
  - "sum(x,P)" means "x is a sum of (the things in) P".

Tarski (1956) (see Betti and Loeb 2012 "On Tarski's Foundations of the Geometry of Solids", *The Bulletin of Symbolic Logic*):

- Definition I. An individual *X* is called a proper part of an individual *Y* if *X* is a part of *Y* and *X* is not identical with *Y*.
- Definition II. An individual *X* is said to be disjoint from an individual *Y* if no individual *Z* is part of both *X* and *Y* .
- Definition III. An individual *X* is called a sum of all elements of a class *a* of individuals if every element of *a* is a part of *X* and if no part of *X* is disjoint from all elements of *a*. ([Tarski, 1956a], p. 25)
- PostulateI. If*X* isapartof*Y* and*Y* isapartof*Z*,then*X* isa part of *Z*.
- Postulate II. For every non-empty class *a* of individuals there ex- ists exactly one individual *X* which is the sum of all elements of *a*.([Tarski, 1956a], p. 25)

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- Core axioms and concepts
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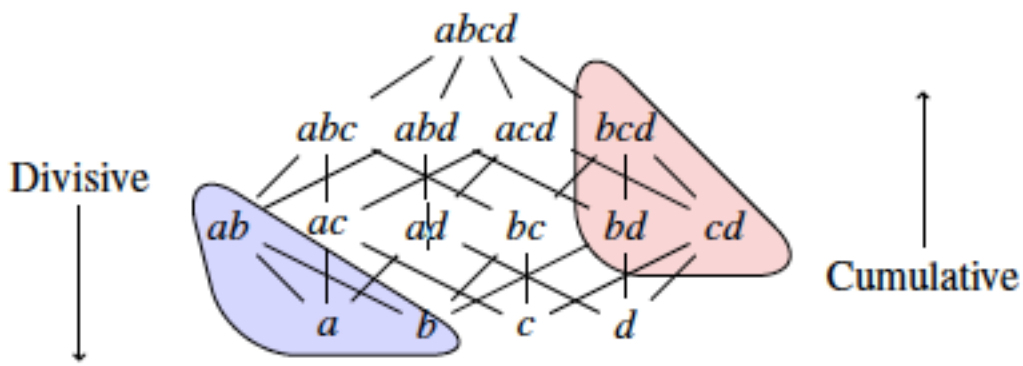
- **Higher order properties:**
  - **cumulativity**
  - **divisivity (aka divisiveness)**
  - **atomicity**

CLASSICAL EXTENSIONAL MEREOLOGY

Cumulativity and Divisivity as Closure Properties



(The representation taken from Grimm 2012, p. 113, Figure 4.4)

- In terms of the part structure, cumulativity is closure under sum formation, while divisivity is closure under part-taking.

- If a predicate is cumulative, it permits "going upwards" in the semilattice, and if it is divisive, it permits "going downwards" in the semilattice.

CLASSICAL EXTENSIONAL MEREOLOGY

Cumulativity

- CUMULATIVE(P) $\leftrightarrow \forall x,y[P(x) \wedge P(y) \rightarrow P(x \oplus y)]$
  A predicate *P* is *cumulative* if and only if, whenever *P* applies to any *x* and *y*, it also applies to the sum of *x* and *y* (assuming that *x* and *y* to which *P* applies are two distinct entities).

- Mass nouns have the property of CUMULATIVE REFERENCE, as Quine (1960, p. 91) proposes: "any sum of parts which are water is water." (Quine attributes this property to Goodman (1951).)

- Holds for true mass nouns (*water*), aggregate mass nouns *furniture*) (Quine (1960) and bare plurals (*apples*) (Link 1983):

(1) A is *water* and B is *water*. $\Rightarrow$ A and B together are *water*.
(2) A are *apples* and B are *apples*. $\Rightarrow$ A and B together are *apples*.

A  $\oplus$ B  $\Rightarrow$ $A \oplus B$ 

- Does not hold for singular count nouns (*boy, apple*):

(3) A is an *apple* and B is an *apple*. $\not\Rightarrow$ A and B together are an *apple*.

A  $\oplus$ B  $\not\Rightarrow$ $A \oplus B$ 

CLASSICAL EXTENSIONAL MEREOLOGY

Divisivity (aka divisiveness)

- Count (sortal) terms are never divisive in their reference, while mass terms may be.

- Mereological definition of DIVISIVE predicates:
  DIVISIVE(P) $\leftrightarrow$ $\forall x \forall y\ [P(x) \wedge y \leq x \ \rightarrow P(y)]$ Krifka 2013 (and elsewhere)
  A predicate *P* is *divisive* if and only if, whenever *P* applies to *x*, then for all *y* such that *y* is a part of *x*, *P* applies to *y*. Or, simpler: If *P* applies to *x*, and if *y* is a part of *x*, then *P* applies to *y*.

- Some notable precursors:
  - sortal vs. non-sortal terms in Frege 1884, p.66 (cited in Pelletier 1975, p.453): "Only a concept which isolates what falls under it in a definite manner, and which does not permit any arbitrary division of it into parts, can be a unit relative to a finite number.
  - *Cheng's condition* (Cheng 1973): "any part of the whole of the mass object which is *W* is *W*" (see Bunt 1979)

- The divisivity property is often assumed to hold for prototypical mass nouns like *water* (this is controversial, see already Quine 1960, p.98).
- The divisivity property does not hold for singular count nouns (*apple, boy*), plurals (*apples*) and for aggregate/collective mass nouns (*furniture*).

CLASSICAL EXTENSIONAL MEREOLOGY

Atomicity

- The property of atomicity characterizes discrete individuals.
  An atom is an individual which has no proper parts:

  $\text{Atom}(x) \leftrightarrow \neg\exists y(y < x)$
  An atom is an individual which has no proper parts.

- Atomicity is a restriction on the part relation. It differs from cumulativity and divisivity in so far as it is not a closure condition.

- Some approaches have models that are atomistic (Link 1983, Chierchia 1998).
  I.e., they have an additional axiom requiring for everything in the domain to be composed of atoms:

  $\forall x\exists y[y \leq x \wedge \neg\exists(z < y)]$ Atomicity
  For any element, there is a part for which there does not exist a proper part.

CLASSICAL EXTENSIONAL MEREOLOGY

Atomicity

- Atoms are also defined relative to a property:

  Atomic(x,P) = $P(x) \wedge \neg\exists y[y < x \wedge P(y)]$ (Atomic relative to a property)
  P applies to *x*, but not to a proper part of *x*.

- Given this definition, we can define what it means for a predicate to be atomic (taken from Krifka 1989):

  Atomic(P) = $\forall x[P(x) \rightarrow \exists y[y \leq x \wedge \text{Atomic}(y, P)]]$ (Atomic predicate)
  P is atomic iff every *x* that is *P* contains a *P*-atom.
  "Atomic(P)" means "atomic relative to a predicate P".

- Singular count nouns (*cat*), bare plurals (*cats*), aggregate mass nouns (*furniture*) express atomic predicates.
- Sometimes also assumed for true mass nouns (*water*), e.g. Chierchia (1998).