Introduction to Formal Language Theory — day 2

Regular Languages

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Outline

Formal languages

Recall: basic definitions

- **alphabet** Σ: nonempty, finite set of **symbols**
- **word** w: a finite string $x_1 \ldots x_n$ of symbols; $(x_1 \ldots x_n \in \Sigma)$
- **length** of a word $|w|$: number of symbols of a word w (example: $|abbaca| = 6$)
- **empty word** ϵ: the word of length 0
- Σ^* is the set of all words over Σ; $(\epsilon \in \Sigma^*)$
- Σ^+ is the set of all nonempty words over Σ $(\Sigma^+ = \Sigma^* \setminus \{\epsilon\})$

Definition

A **formal language** L is a set of words over an alphabet Σ, i.e. $L \subseteq \Sigma^*$.

Type3-languages / right-linear languages

- the class of Type 3 languages can be generated by **right-linear grammars**

Definition

A grammar (N, T, S, R) is **Type3** or **right-linear** iff all rules are of the form:

$$A \to a \text{ or } A \to wB \text{ with } A, B \in N, a \in T, \text{ and } w \in T^*$$

Additionally, the rule $S \to \epsilon$ is allowed iff S does not appear in any right-hand side of a rule.

A language generated by a right-linear grammar is said to be a **right-linear language** or a **Type3-language**.

[Remember, we write $L(G)$ for the language generated by a grammar G.]

Type3-languages / right-linear languages

Examples:

- $P = \{S \to aB, B \to bB, B \to bA, A \to a\}$
  generates (ab^*a)
- $P = \{S \to \epsilon, S \to aA, S \to bB, A \to aA, A \to \epsilon, A \to bB, B \to bB, B \to \epsilon\}$
  generates (a^*b^*)

Regular expressions

- the class of Type 3 languages can be described by **regular expressions**

The set of **regular expressions** *$RegEx_\Sigma$* over an alphabet $\Sigma = \{a_1, \ldots, a_n\}$ is defined by:

- $\emptyset$ is a regular expression.
- ϵ is a regular expression.
- $a_1, \ldots, a_n$ are regular expressions.
- If a and b are regular expressions over Σ then
  - $(a|b)$
  - ab
  - $a^\star$

  are regular expressions too.

Regular expressions

RegEx: semantics

Each regular expression r over an alphabet Σ denotes a formal language $L(r) \subseteq \Sigma^*$.

Regular languages are those formal languages which can be expressed by a regular expression.

The denotation function L is defined inductively:

- $L(\emptyset) = \emptyset$, $L(\epsilon) = \{\epsilon\}$, $L(a_i) = \{a_i\}$
- $L(r_1|r_2) = L(r_1) \cup L(r_2)$
- $L(r_1 r_2) = L(r_1) \frown L(r_2)$
- $L(r^*) = L(r)^*$

'r^+' is used as a short-hand for '$r \frown r^*$'.

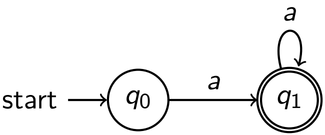
Examples: regular expressions

Find a regular expression which describes the regular language L (be careful: at least one language is not regular!)

- L is the language over the alphabet $\{a, b\}$ with $L = \{aa, \epsilon, ab, bb\}$. $aa|\epsilon|ab|bb$
- L is the language over the alphabet $\{a, b\}$ which consists of all words which start with a nonempty string of a's followed by any number of b's. a^+b^*
- L is the language over the alphabet $\{a, b\}$ such that every a has a b immediately to the right. $b^*(ab^+)^*$
- L is the language over the alphabet $\{a, b\}$ which consists of all words which contain an even number of a's. $b^*(ab^*a)^*b^*$
- L is the language of all palindromes over the alphabet $\{a, b\}$. not regular!

Deterministic finite-state automaton (detFSA)

- the class of Type 3 languages can be accepted (recognized) by **deterministic finite-state machines** (detFSA)
- example: detFSA for the language $L(a^+)$



- initial state q_0, final state q_1
- transitions from q_0 to q_1 reading an a, from q_1 to q_1 reading an a

Deterministic finite-state automaton (detFSA)

Definition

*A **deterministic finite-state automaton** is a 5-tuple* $(Q, \Sigma, \delta, q_0, F)$ *with:*

1. *a finite, nonempty set of **states*** Q
2. *an alphabet* Σ *with* $Q \cap \Sigma = \emptyset$
3. *a **transition function*** $\delta : Q \times \Sigma \to Q$
4. *an **initial state*** $q_0 \in Q$ *and*
5. *a set of **final states*** $F \subseteq Q$

Language accepted by an automaton

Definition

*A **situation** of a finite-state automaton* $(Q, \Sigma, \delta, q_0, F)$ *is a triple* (x, q, y) *with* $x, y \in \Sigma^*$ *and* $q \in Q$.
Situation (x, q, y) ***produces*** *situation* (x', q', y') ***in one step*** *if there exists an* $a \in \Sigma$ *such that* $x' = xa$, $y = ay'$ *and* $\delta(q, a) = q'$, *we write* $(x, q, y) \mapsto (x', q', y')$ *[*$(x, q, y) \mapsto^* (x', q', y')$ *as usual].*

Definition

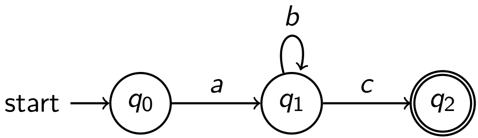
A word $w \in \Sigma^*$ *gets **accepted** by an automaton* $(Q, \Sigma, \delta, q_0, F)$ *if* $(\epsilon, q_0, w) \mapsto^* (w, q_n, \epsilon)$ *with* $q_n \in F$.
An automaton accepts a language iff it accepts every word of the language. We write $L(A)$ *for the language accepted by an automaton* A.

Examples (detFSA)

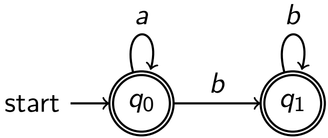
- accepting the language $L(ab^*c)$

  $M_1 = (\{q_0, q_1, q_2\}, \{a, b, c\}, \{(q_0, a, q_1), (q_1, b, q_1), (q_1, c, q_2)\}, q_0, \{q_2\})$

  state diagram:

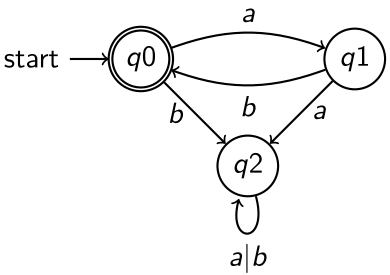


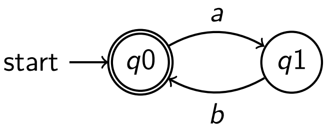
- accepting the language a^*b^*



$M_1 = (\{q_0, q_1\}, \{a, b\}, \{(q_0, a, q_0), (q_0, b, q_1), (q_1, b, q_1)\}, q_0, \{q_0, q_1\})$

Example

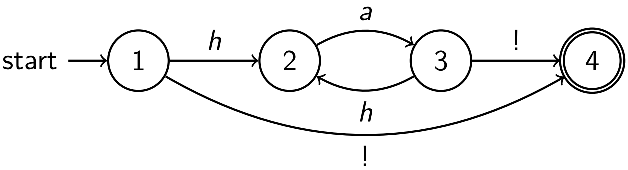


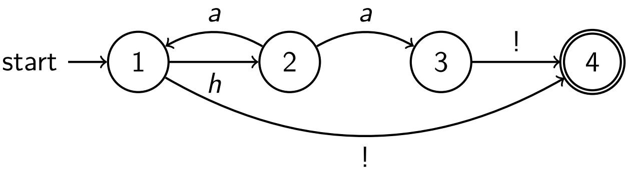


- both automatons accept language $L((ab)^*)$
- in automaton graphs we often omit the trap state (partial transition function)

Examples

- accepting the language $(ha)^*!$





Nondeterministic finite-state automaton (nondetFSA)

Definition

*A **nondeterministic finite-state automaton** is a 5-tuple* $(Q, \Sigma, \Delta, q_0, F)$ *with:*

1. *a finite nonempty set of **states*** Q
2. *an alphabet* Σ *with* $Q \cap \Sigma = \emptyset$
3. **a transition relation** $\Delta \subseteq Q \times \Sigma \times Q$
4. *an **initial state*** $q_0 \in Q$ *and*
5. *a set of **final states*** $F \subseteq Q$

nondetFSA: extensions

- an **ϵ-transition** $\xrightarrow{\epsilon}$ allows to change the state without reading a symbol
- a **regular-expression transition** $\xrightarrow{r}$ allows to change the state by reading in any string $s \in L(r)$

Equivalence of detFSA and nondetFSA

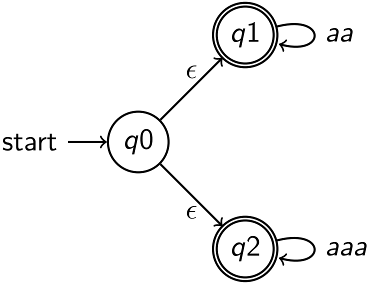
Theorem of Rabin & Scott

A language L is accepted by a detFSA iff L is accepted by a nondetFSA (with ϵ-transitions and/or regular-expression transitions).

- Why is it useful to have both notions?
  - the detFSAs are conceptually more straightforward
  - it is often easier to construct a nondetFSA
  - for some other classes of automata the two subclasses are not equivalent, so the notions remain important
- example:
  - $L : \{a^n \mid n \text{ is even or dividable by } 3\}$ (or $\mathrm{L}((aa)^* \mid (aaa)^*)$)
  - $\mathrm{L}((aa)^* \mid (aaa)^*)$ is accepted by the automata on the following slides: regex-FSA, ϵ-FSA, nondetFSA and detFSA

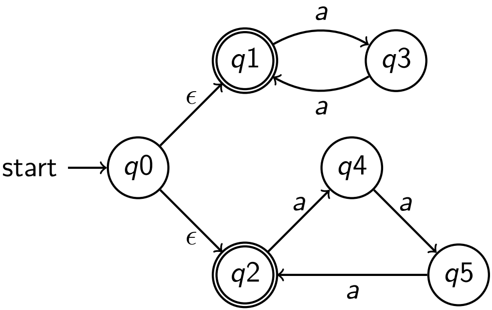
Equivalence of detFSA and nondetFSA

- L($(aa)^* \mid (aaa)^*$) with regex-FSA



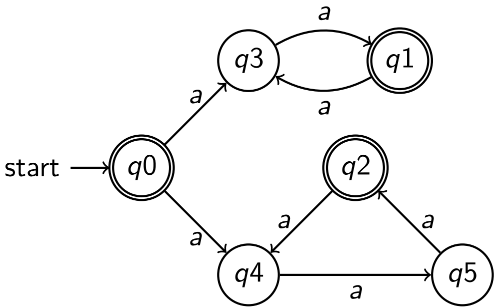
Equivalence of detFSA and nondetFSA

- L($(aa)^* \mid (aaa)^*$) with ϵ-FSA



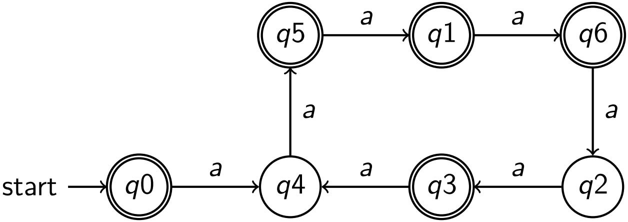
Equivalence of detFSA and nondetFSA

- L($(aa)^* \mid (aaa)^*$) with nondetFSA



Equivalence of detFSA and detFSA

- L($(aa)^* \mid (aaa)^*$) with detFSA

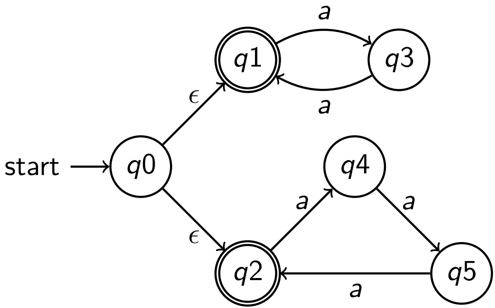


Eliminating ϵ-transitions

- the ϵ-closure of a state q (denoted as $ECL(q)$) is the set that contains q together with all states that can be reached starting at q by following only ϵ-transitions
- Given an ϵ-FSA M eliminating ϵ-transitions produces an nondetFSA M' such that $L(M') = L(M)$.
- The construction of M' begins with M as input, and takes 3 steps:
  1. Make q an accepting state iff $ECL(q)$ contains an accepting state in M.
  2. Add an arc from q to q' labeled a iff there is an arc labeled a in M from some state in $ECL(q)$ to q'.
  3. Delete all arcs labeled ϵ.

Eliminating ϵ-transitions

- the the ϵ-FSA for $L((aa)^* \mid (aaa)^*)$

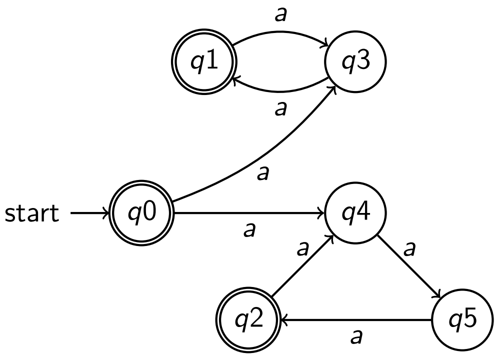


- $ECL(q_0) = \{q_0, q_1, q_2\}$

1. make q_0 an accepting (final) state
2. add the arcs: from q_0 to q_3 by a and q_0 to q_4 by a
3. Delete all arcs labeled ϵ.

Eliminating ϵ-transitions

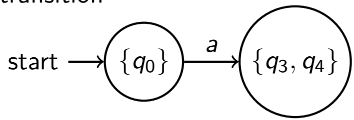
- step 1 – 2. resulting in:



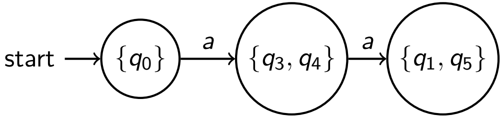
- nondetFSA (see slide 19)

nondetFSA to detFSA

- make the nondetFSA from the previous slide deterministic
- remove multiple transitions with the same symbol
- idea: each state in detFSA will be a **set of states** from the nondetFSA
  - from q_0 we can go with a to q_3 and q_4
    $\Rightarrow$ in the detFSA we have the states $\{q_0\}$ and $\{q_3, q_4\}$ with an a transition

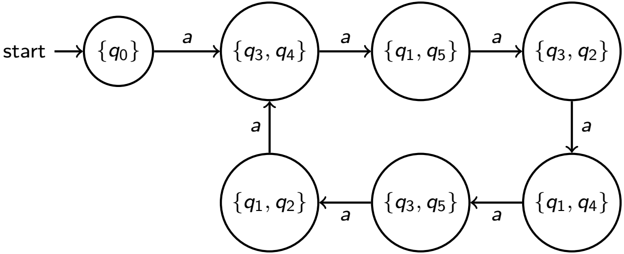


  - from the states in $\{q_3, q_4\}$ we can go with a to q_1 and q_5
    $\Rightarrow$ in the detFSA we add the state $\{q_1, q_5\}$ with an a transition from $\{q_3, q_4\}$

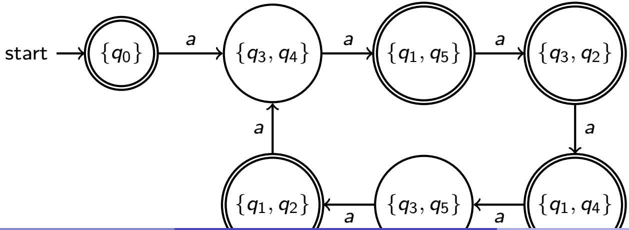


nondetFSA to detFSA

- repeat the steps as before, result in in:



- make all states final, where any of the states in the set were final states in the nondetFSA



Theorem of Kleene

Theorem

If L is a formal language, the following statements are equivalent:

- *L is regular (i.e., describable by a regular expression)*
- *L is right-linear (i.e., generated by a right-linear grammar)*
- *L is FSA-acceptable (i.e., accepted by a finite state automaton)*

Proof idea:

1. every regular language is right-linear
2. every right-linear language is FSA-acceptable
3. every FSA-acceptable language is regular

Proof: Every regular language is right-linear

$\Sigma = \{a_1, \ldots, a_n\}$

1. $L(\emptyset)$ is generated by $(\{S\}, \Sigma, S, \{\})$,
2. $L(\epsilon)$ is generated by $(\{S\}, \Sigma, S, \{S \to \epsilon\})$,
3. $L(a_i)$ is generated by $(\{S\}, \Sigma, S, \{S \to a_i\})$,
4. If $L(r_1)$, $L(r_2)$ are regular languages described by r_1, r_2 with generating right-linear grammars (N_1, T_1, S_1, P_1), (N_2, T_2, S_2, P_2), then $L(r_1|r_2)$ is generated by $(N_1 \uplus N_2, T_1 \cup T_2, S, P_1 \cup_{\uplus} P_2 \cup \{S \to S_1, S \to S_2\})$,
5. $L(r_1 r_2)$ is generated by $(N_1 \uplus N_2, T_1 \cup T_2, S_1, P_1' \cup_{\uplus} P_2)$ (P_1' is obtained from P_1 if all rules of the form $A \to b$ ($b \in T$) are replaced by $A \to bS_2$),
6. $L(r_1^*)$ is generated by $(N_1, \Sigma, S_1, P_1' \cup \{S_1 \to \epsilon, S_1 \to S\})$ (P_1' is obtained from P_1 if by all rules of the form $A \to b$ ($b \in T$) we add the rule $A \to bS$).

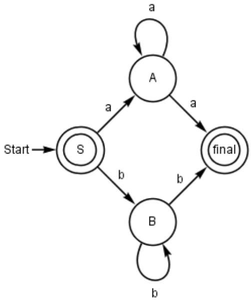
Proof: Every right-linear language is FSA-acceptable

If $G = (N, T, S, R)$ is a right-linear grammar then the nondetFSA $M = (N \cup \{\text{final}\}, T, \Delta, S, F)$ with

- $F = \{\text{final}, S\}$ if $S \to \epsilon \in R$ or else $F = \{\text{final}\}$.
- $(A, a, B) \in \Delta$, if $A \to aB \in R$ and $(A, a, \text{final}) \in \Delta$ if $A \to a \in R$.

accepts $L(G) = L(M)$.

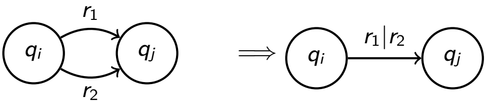
$$S \to aA,\ S \to bB,\ S \to \epsilon,\ A \to aA,\ A \to a,\ B \to bB,\ B \to b$$



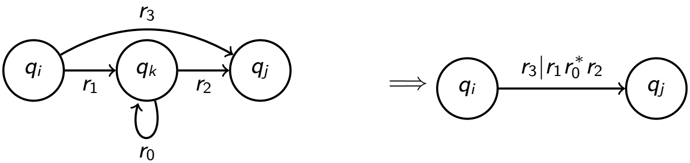
Every FSA-acceptable language is regular

Let $M = (Q, \Sigma, \Delta, q_0, F)$ be a nondetFSA.

1. Construct an equivalent automaton M' with only one final state and no incoming transitions at the start state: $M = (Q \cup \{q_s, q_f\}, \Sigma, \Delta', q_s, \{q_f\})$ with $\Delta' = \Delta \cup \{(q_s, \epsilon, q_0)\} \cup \{(q_i, \epsilon, q_f | q_i \in F\}$.
2. For each pair of states (q_i, q_j) replace all $(q_i, r_1, q_j) \in \Delta', (q_i, r_2, q_j) \in \Delta', \ldots$ by a single transition $(q_i, r_1 | r_2 | \ldots, q_j)$.



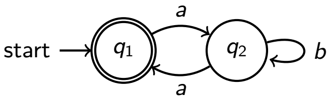
3. As long as there is still a state $q_k \notin \{q_s, q_f\}$ eliminate q_k by the following rule:



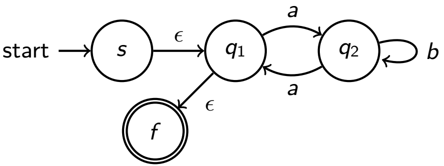
4. Finally the automaton consists only of the two states q_s and q_f and one single transition (q_s, r, q_f) and $L(M) = L(r)$.

Example

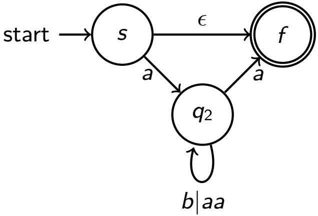
- starting with the FSA:

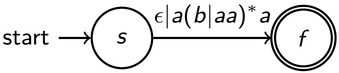


- adding ϵ-transitions:



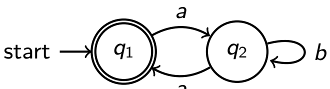
- eliminating q_1:
  - $s \xrightarrow{a} q_2$
  - $s \xrightarrow{\epsilon} f$
  - $q_2 \xrightarrow{a} f$
  - $q_2 \xrightarrow{aa} q_2$



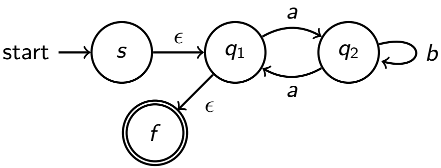


Example

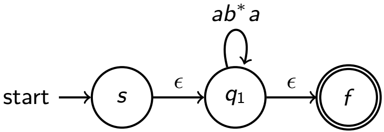
- starting with the FSA:

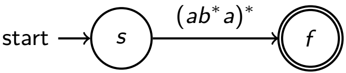


- adding ϵ-transitions:



- eliminating q_2:
  - $q_1 \xrightarrow{ab^*a} q_1$





Intuitive rules for regular languages

- L is regular if it is possible to check the membership of a word simply by reading it symbol by symbol while using only a finite stack.
- Finite-state automatons are too weak for:
  - unlimited counting in $\mathbb{N}$ ("same number as");
  - recognizing a pattern of arbitrary length ("palindrome");
  - expressions with brackets of arbitrary depth.